A SUFFICIENT CONDITION FOR SMOOTHNESS OF SOLUTIONS OF NAVIER-STOKES EQUATIONS

BY
S. KANIEL

ABSTRACT

The main theorem in this paper states that if a certan bound is imposed on the associated pressure pertaining to a weak solution of the Navier-Stokes equation then the solution is actually smooth. The proof uses the fact that such a bound implies a bound on the first derivatives of the solution which, in turn, leads to smoothness.

1. There is a considerable interest in the investigation of sufficient conditions that will insure smoothness of the solutions. Serrin [1] was the first one to establish such a condition. The author, together with M. Shinbrot [2] could strengthen his results. Both conditions involve the flow itself. The purpose of this paper is to furnish a condition that involves the associated pressure.

The work is based on [3]. We use some lemmas and the existence theorems proved there; we also use a similar technique; thus we use the same notations.

2. We shall study the Navier-Stokes equations in the form

$$u_t - \nabla^2 u + u \cdot \nabla u = -\nabla p \tag{2.1}$$

$$\nabla \cdot u = 0 \tag{2.2}$$

with the density and the kinematic viscosity normalized to one. We suppose further that the flow takes place in a bounded, smooth domain $D \subset R^3$. In the above equation, u denotes the velocity of the flow, and p the pressure. In addition to (2.1) and (2.2), the velocity must satisfy the boundary condition

$$u = 0 \text{ for } x \in \partial D \tag{2.3}$$

and an initial condition

$$u = u^0 \text{ for } t = 0. \tag{2.4}$$

Given any vector u, we denote its Euclidean length by $|u|$. If $u = u(x,t)$ we write

$$\| u \|_q = \int_D |u|^q dx^{\,1/q} \tag{2.5}$$

if the right-hand side is well defined.

Received March 28, 1968, and revised form, August 7, 1968.

A weak solution of the Navier-Stokes equations is a vector $u(x,t)$ that satisfies

$$\|u\|_2 \leqq M_1 \qquad 0 \leqq t \leqq T$$

$$\int_0^T \int |\nabla u|^2 dx dt \leqq M_2 .$$

It also satisfies (2.1)–(2.4) in a weak sense, i.e. for any smooth $\phi(x,t)$ that satisfied (2.2), (2.3) the following holds

$$\int_0^T \int \left[-u \frac{\partial \phi}{\partial t} - u \nabla^2 \phi + (u \cdot \nabla \phi) \cdot u \right] dx dt$$
$$= \int u(x,T)\phi(x,T)dx - \int u(x,0)\phi(x,0)dx.$$

THEOREM. *Let u be a weak solution of the Navier-Stokes equations* (2.1)–(2.4), *with smooth initial conditions.*

If p, the associated pressure, satisfies

(2.6) $$\|p\|_q \leqq C_0 \qquad 0 \leqq t \leqq T_1$$

for some $q > 12/5$ *then u is smooth.*

Proof. We shall prove that u is a strong solution of (2.1)–(2.4). This means that all the terms in (2.1) are well defined in L_2. From this point, the passage to a smooth solution is done by the aid of Theorem 7.1 of [2].

It is known by Theorem 5.1 of [3] (also by Prodi's work [4]) that for any initial conditions u^0 so that $\|\nabla u^0\|_2$ is bounded there exists an interval $0 \leqq t \leqq T$ for which the solution u is strong. Moreover T is dependent on $\|\nabla u^0\|$ alone. We shall prove that if (2.6) holds, then:

(2.7) $$\|\nabla u\|_2 \leqq k(C_0, \|\nabla u^0\|)$$

for all t for which $u(x,t)$ is strong.

Thus, given a subinterval $[0, T_2]$ for which u is strong we can extend it using (2.7) to an interval $[0, T_2 + T(k(C_0, \|\nabla u^0\|_2))]$; covering the interval $[0, T_1]$ in a finite number of steps. In order to get (2.7) we proceed as follows.

Let the subinterval $[0, T_2]$ be given. Multiply (2.1) by the vector $u^3 = (u_1^3, u_2^3, u_3^3)$ and integrate over D.

(2.8) $$\int u_t \cdot u^3 dx - \int \nabla^2 u \cdot u^3 dx + \int (u \cdot \nabla u) \cdot u^3 dx = - \int \nabla p \cdot u^3 dx.$$

The first term is rewritten as:

$$\int u_t \cdot u^3 dx = \frac{1}{4}\frac{d}{dt}\| u^2 \|_2^2. \tag{2.9}$$

Integrate by parts to get:

$$\int (u \cdot \nabla u) u^3 dx = 0 \tag{2.10}$$

$$-\int \nabla^2 u \cdot u^3 dx = \int \nabla u \cdot \nabla u^3 dx = \frac{3}{4}\| \nabla u^2 \|_2^2. \tag{2.11}$$

Denote the index q in (2.6) as $q = \dfrac{12}{5 - 5\varepsilon.}$.

Then:

$$\left| \int \nabla p \cdot u^3 dx \right| = \left| \int p \cdot (\nabla \cdot u^3) dx \right| \leqq \frac{3}{2} \left| \int (\nabla \cdot u^2) \cdot |u| \cdot |p| dx \right.$$

$$\leqq \frac{3}{2}\| \nabla u^2 \|_2 \int |u|^2 p^2 dx^{1/2} \leqq \frac{3}{2}\| \nabla u^2 \|_2 \| u^2 \|_{6/1+5\varepsilon} \| p^2 \|_{6/5-5\varepsilon}$$

$$\leqq \frac{3}{2}\| \nabla u^2 \|_2 \| u^2 \|_6^{1-\varepsilon} \| u^2 \|_1^{\varepsilon} \| p \|_q^2.$$

Since $u = 0$ on ∂D it follows, by Sobolev's lemma, that

$$\| u^2 \|_6 \leqq C \| \nabla u^2 \|_2.$$

Since u is a weak solution of (2.1)–(2.4) it satisfies [5].

$$\| u^2 \|_1 = \| u \|_2^2 \leqq C. \tag{2.12}$$

Substituting (2.9)–(2.12) in (2.8) we get:

$$\frac{d}{dt}\| u \|_2^2 + 3 \| \nabla u^2 \|_2^2 \leqq C \cdot C_0^2 \| \nabla u^2 \|_2^{2-\varepsilon}.$$

Since u is smooth in $[0, T_2]$ it follows that $(d/dt)\| u^2 \|_2^2$ exists and is continuous. Consider now the closed set E defined by

$$\frac{d}{dt}\| u^2 \|_2^2 \leqq 0 \qquad t \in E.$$

For $t \in E$ the following holds:

$$\| \nabla u^2 \|_2 \leqq \left\{ \frac{C \cdot C_0}{3} \right\}^{1/\varepsilon}.$$

Consequently, by Poincaré's inequality:

$$\| u^2 \|_2 \leqq C \cdot \left\{ \frac{C \cdot C_0^2}{3} \right\}^{1/\varepsilon} = k(C_0). \tag{2.13}$$

The complementary set consists of possibly a half open interval $[0, \bar{t}_0]$ and a a sequence of open intervals $(t_i, \bar{t}_i)$. Since $\|u^2\|_2$ is decreasing in any of these sub-intervals, and since the points t_i belong to E we get:

$$\|u^2\|_2 \leqq \|(u^0)^2\|_2 \leqq C\|\nabla u^0\|_2^2 \qquad 0 \leqq t \leqq \bar{t}_0.$$

$$\|u^2(t)\|_2 \leqq \|u^2(t_i)_2\| \leqq k(C_0) \qquad t < t < \bar{t}_i.$$

Thus $\|u_2^2\|_2 = \|u\|_4^2$ is uniformly bounded in $[0, T_2]$.

Now consider the space L_2, space of all vectors having components in L_2 and consider the projection P onto the solenoidal vectors. As in the proof of Theorem (4.1) in [3], multiply equation (2.1) by $-P\nabla^2 u$. Inequality (4.8) of the latter reads

$$\frac{d}{dt}\|\nabla u\|_2^2 + \|P\nabla^2 u\|_2^2 \leqq \|u \cdot \nabla u\|_2^2. \tag{2.14}$$

Now we proceed as follows:

$$\|u \cdot \nabla u\|_2^2 \leqq \|u\|_4^2 \|\nabla u\|_4^2 \leqq \|u\|_4^2 \|\nabla u\|_2^{1/2} \|\nabla u\|_6^{3/2}.$$

Lemmas 3.3 and 3.5 of [3] read:

$$\|\nabla u\|_6 \leqq C\|P\nabla^2 u\|_2$$

$$\|\nabla u\|_2 \leqq \|u\|_2^{1/2} \|\nabla P^2 u\|_2^{1/2}.$$

Substituting the last two inequalities we get

$$\|u \cdot \nabla u\|_2^2 \leqq C\|P\nabla^2 u\|_2^{7/4} \|u\|_4^2 \|u\|_2^{1/4} \leqq C\|P\nabla^2 u\|_2^{7/4}$$

Thus

$$\frac{dt}{d}\|\nabla u\|_2^2 + \|P\nabla^2 u\|_2^2 \leqq C\|P\nabla^2 u\|_2^{7/4}. \tag{2.15}$$

Let us repeat the argument that was used in the first step. Let F be the closed set for which

$$\frac{d}{dt}\|\nabla u\|_2^2 \leqq 0 \qquad t \in F.$$

For $t \in F$ $P\nabla^2 u$ is uniformly bounded, therefore by Lemma 3.4 of [3] it follows that

$$\|\nabla u\|_2 \leqq C \qquad t \in F. \tag{2.16}$$

The complementary set consists of possibly the half open interval $[0, \bar{t}_0]$ for which

$$\|\nabla u(t)\|_2 \leqq \|\nabla u^0\|_2 \qquad 0 \leqq t < \bar{t}_0, \tag{2.17}$$

and intervals $(t_i, \bar{t}_i)$ for which

$$\|\nabla u(t)\|_2 \leqq \|\nabla u(t_i)\|_2 \leqq C \qquad t_i < t < \hat{t}_i \tag{2.18}$$

since $t_i \in F$.

Estimates (2.16)–(2.18) are summarized in (2.7). Thus it is possible to extend the domain smoothness across T_2 and the proof is complete.

References

1. J. A. Serrin, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Archs. Nation. Mech. Analysis, **9** (1962), 187–195.

2. S. Kaniel and M. Shinbrot, *Smoothness of weak solutions of the Navier-Stokes equations*, Archs. Nation. Mech. Analysis, **24** (1967), 302–324.

3. M. Shinbrot and S. Kaniel, *The initial value problem for the Navier-Stokes equations*, Archs. Nation. Mech. Analyses, **21** (1966), 270–285.

4. G. Prodi, *Theoremi di tipo locale per il sistema di Navier-Stokes e stabilita delle soluzioni stazionairie*, Rc. Semin. Mat. Univ. Padova, **32** (1962), 374–397.

5. Hopf, Eberhardt, *Über die Anfangwertaufgaben für die hydroamischen Grundgleichungen*, Math. Nach., **4** (1951), 213–321.

The Hebrew University of Jerusalem